

# Definitions and Propositions

## Part A: Order Theory 1

Partially ordered sets (posets) are the central notion of order theory.

### Definition A.1 - Poset

A binary relation  $\leq$  on a set  $A$  is called a *partial order*, if for all  $x, y, z \in A$  the following conditions hold:

(Reflexivity)	$x \leq x$
(Antisymmetry)	$x \leq y$ and $y \leq x \Rightarrow x = y$
(Transitivity)	$x \leq y$ and $y \leq z \Rightarrow x \leq z$

$\mathbf{A} = (A, \leq)$  is called a "*partially ordered set*", in short, a *poset*.

### Definition A.2 - Extreme Elements, Covers (Neighbors)

$\mathbf{A} = (A, \leq)$  is a poset,  $B \subseteq A$  a subset of  $A$ . A *maximal element* of  $B$  is an element  $b \in B$  where  $b < a \Rightarrow a \notin B$  for all  $a \in A$ . The *maximum* of  $B$  is an element  $b \in B$  with  $b' \leq b$  for all  $b' \in B$ .

A *minimal element* of  $B$  is an element  $b \in B$  where  $a < b \Rightarrow a \notin B$  for all  $a \in A$ . The *minimum* of  $B$  is an element  $b \in B$  with  $b \leq b'$  for all  $b' \in B$ .

$b$  is called *upper cover (neighbor)* (or *lower cover* of  $b$ ), if  $a \leq b$  and if there is no  $x \neq a, b$  where  $a \leq x \leq b$ .

Posets  $\mathbf{A} = (A, \leq)$  can be visualised with the help of so called *Hasse diagrams*. Elements of  $A$  are represented by points or small circles. The line segments between the points reflect the order relation.  $b$  is drawn above  $a$  and is connected to  $a$  by a line segment if and only if  $b$  is the upper cover of  $a$ . A Hasse diagram stores all information of a poset. Hasse diagrams are not unambiguously defined, i.e. a poset possesses several possible Hasse diagrams (cf. figure A).

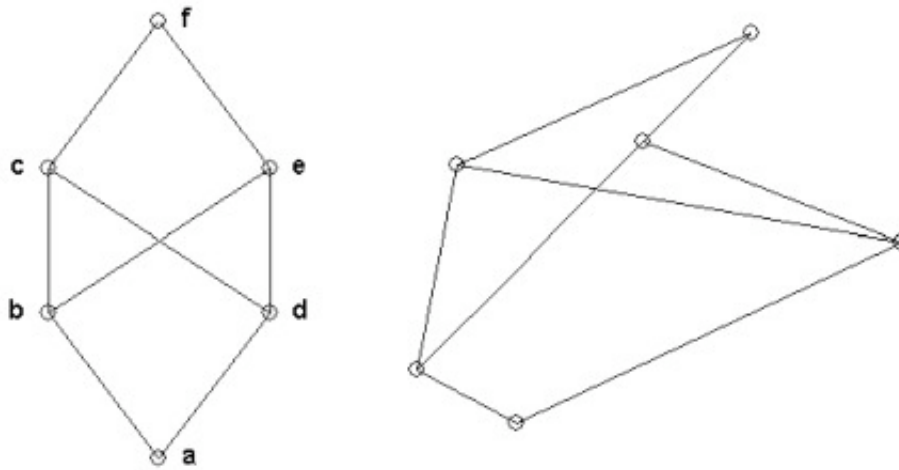


Figure A: Two Hasse diagrams of the same posets

### Remark A.3 - Interpretation of Partial Order $\leq$

In the definitions, propositions, and examples used on this site we will henceforth use the standard interpretation of the order relation  $\leq$  and all its related relations, such as  $<$ ,  $\geq$ ,  $>$ ,...

In a poset not all elements need to be ordered. This means that it is possible that the set  $A$  contains two elements which cannot be ordered, cannot be compared.

In the following, this fact and other fundamental notions concerning posets will be shown in a slightly more mathematical fashion.

### Definition A.4 - Comparable Elements

$\mathbf{A} = (A, \leq)$  is a poset. Two elements  $a, b \in A$  are called *comparable*, if  $a \leq b$  or  $b \leq a$ . Else they are called *incomparable*.

### Definition A.5 - Chain

A *chain* is a poset in which any two elements are comparable. The poset  $\mathbf{C}_n := (C, \leq)$  is called an *n-chain*, if  $C$  contains  $n$  elements and all of them form a chain under  $\leq$ .

### Definition A.6 - Subposet, Order-Homomorphism, Order-Embedding, Order-Isomorphism

A poset  $(B, \leq_B)$  is a *subposet* of the poset  $(A, \leq_A)$  if  $B \subseteq A$  and  $\leq_B \subseteq \leq_A$ , i.e. for all  $x, y \in B$   $x \leq_B y$  if and only if  $x \leq_A y$ .

If  $(A, \leq_A)$  and  $(B, \leq_B)$  are posets, then a map  $f : A \rightarrow B$  is called *order-preserving* or an *order-homomorphism* if  $x \leq_A y$  implies  $f(x) \leq_B f(y)$  for all  $x, y \in A$ .

If additionally  $f$  is injective and  $f(x) \leq_B f(y)$  implies  $x \leq_A y$  for all  $x, y \in A$ , then  $f$  is called an *order-embedding*.

Surjective order-embeddings are called *order-isomorphisms*.

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The computer program "Algebra Workbench" (AWB) was created by Markus Sprenger. The documentation found here is based on a 2005 [master thesis](#) by Christoph Röthlisberger. The translation and adaptation of the material was done by Cindy-Jane Armbruster.

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