

## Definitions and Propositions

### Part A: Order Theory 2

#### Definition A.7 - Bounds, Infimum, Supremum

$\mathbf{A} = (A, \leq)$  is a poset,  $B \subseteq A$  is a subset of  $A$ . An element  $a \in A$  is called *upper bound* of  $B$ , if  $b \leq a$  for all  $b \in B$ . An upper bound  $a$  of  $B$  is called *least upper bound* of  $B$  or *supremum* of  $B$  ( $\sup B$ ), if  $a \leq a'$  for all upper bounds  $a'$  of  $B$ .

An element  $a \in A$  is called *lower bound* of  $B$ , if  $b \geq a$  for all  $b \in B$ . A lower bound  $a$  of  $B$  is called *greatest lower bound* of  $B$  or *infimum* of  $B$  ( $\inf B$ ), if  $a \geq a'$  for all lower bounds  $a'$  of  $B$ .

The notation  $x \vee y$  ("*x join y*") for  $\sup\{x, y\}$ , and  $x \wedge y$  ("*x meet y*") for  $\inf\{x, y\}$  is often used. (Cf. also [meet in congruence class](#))

#### Remark A.8 - algebraizable

In the AWB an order is called *algebraizable* if and only if at least one of the operations meet or join is totally defined.

To illustrate [Definition A.5 - Chain](#) we have here a few remarks regarding the poset  $\mathbf{A} = (A, \leq)$  which is visualised in figure A below:

$\{a, b, c\} \subseteq A$  forms a chain. Elements  $b$  and  $d$  are [incomparable](#), while elements  $a$  and  $f$ , for example, are comparable. The subset  $\{a, b, c, d, e\}$  contains no [maximum](#), but instead there are two maximal elements:  $c$  and  $e$ . The elements  $c, e,$  and  $f$  are [upper bounds](#) of this set. Because there is no least upper bound the set  $\{a, b, c, d, e\}$  contains no [supremum](#). (It does have an infimum:  $a$ ). In  $\mathbf{A}$  the element  $e$  possesses exactly one [upper cover](#): element  $f$ .

Next to posets, lattices are another important construct; especially, but not only, in order theory.

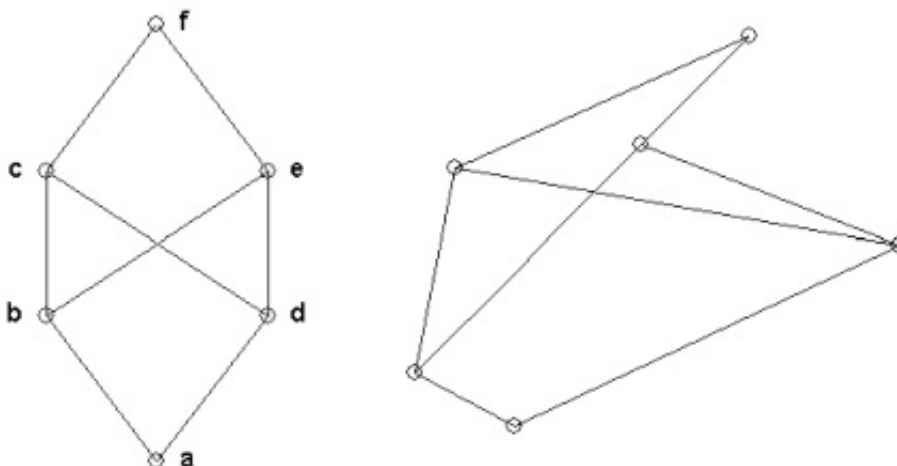


Figure A: Two Hasse diagrams of the same posets

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**Definition A.9 - Lattice** (cf. also [Definition B.2 - Lattice](#))

A *lattice* is a poset  $\mathbf{A} = (A, \leq)$  for which the following holds: For all  $x, y \in L$  there exists the supremum  $\sup\{x, y\}$  and the infimum  $\inf\{x, y\}$ .

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**Definition A.10 - Complete Lattice**

A lattice  $\mathbf{A} = (A, \leq)$  is called *complete*, if and only if any arbitrary subset  $X \subseteq A$  contains a supremum and an infimum.

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**Remark A.11**

According to [Definition A.10 - Complete Lattice](#) every finite lattice is complete.

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The computer program "Algebra Workbench" (AWB) was created by Markus Sprenger. The documentation found here is based on a 2005 [master thesis](#) by Christoph Röthlisberger. The translation and adaptation of the material was done by Cindy-Jane Armbruster.

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