Definitions and Propositions

Part A: Order Theory 3

The question now is, how to find, for any arbitrary poset (X,P), the smallest possible complete lattice into which to embed the poset. The *Dedekind-MacNeille Completion* is the answer to that question. Before we can illustrate how the completion of a poset works, we need to introduce some central notions.

Definition A.12 - Down-Set, Up-Set, A^L, A^U

(X,P) is a poset, $A \subseteq X$ is a subset. We define:

- $\downarrow A = \{x \in X : \exists a \in A \text{ with } x \leq a\}$ (Down-Set of A)
- $\uparrow A = \{x \in X : \exists a \in A \text{ mit } a \leq x\}$ (Up-Set of A)
- $\downarrow x = \downarrow\{x\}$
- $\uparrow x = \uparrow\{x\}$
- $A^L = \{x \in X : x \leq a \ \forall a \in A\}$ (Set of lower bounds of A)
- $A^U = \{x \in X : a \leq x \ \forall a \in A\}$ (Set of upper bounds of A)

We are looking for the smallest possible lattice into which we can fit our poset. This means that the poset needs to be dense in the lattice. In mathematical terms:

Definition A.13 - Dense

(X, P) is a poset. $S \subseteq X$ is called $\wedge$-*dense* (or $\vee$-*dense*), if and only if for all $x \in X$ there exists a $T \subseteq S$ where $\inf T = x$ (or $\sup T = x$).

If S is $\wedge$-dense as well as $\vee$-dense, then S is *dense*.

Definition A.14 - (Γ_{UL})

(X, P) is a poset. The (closure-)operator Γ_{UL} is defined as follows:

$$\Gamma_{UL}: A \subseteq X \mapsto (A^U)^L$$

Proposition A.15 - Dedekind-MacNeille (cf. also Dedekind-MacNeille Completion)

(X, P) is a poset and $DM(X, P) = (im(\Gamma_{UL}), \subseteq)$. $\phi: X \rightarrow \wp(X)$ is given by $x \mapsto \downarrow x$.
Then, the following holds:

a. DM(X, P) is a complete lattice
b. ϕ is an order embedding
c. If (X, P) is already a complete lattice, then $(X, P) \cong DM(X, P)$, especially ϕ is an isomorphism of lattices.
d. $\phi(X)$ is dense in DM(X, P)

Remark A.16

The poset DM(X, P) = (im(Γ_{UL}), $\subseteq$) together with the embedding ϕx is called Dedekind-MacNeille Completion of the poset (X, P). This is the wanted smallest complete lattice which contains the poset (X, P).

Cf. also the illustration in Part C: Dedekind-MacNeille Completion.

The computer program "Algebra Workbench" (AWB) was created by Markus Sprenger. The documentation found here is based on a 2005 master thesis by Christoph Röthlisberger. The translation and adaptation of the material was done by Cindy-Jane Armbruster.

This page was designed by cja in 2006. It was last updated on September 10, 2006.